A NEW CHARACTERIZATION OF EBERLEIN COMPACTA

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L. ONCINA

ABSTRACT. In this paper we give a sufficient and necessary condition for a Radon-Nikodým compact space to be Eberlein compact in terms of a separable fibre connecting weak-* and norm approximation.

INTRODUCTION

A compact topological space is called Eberlein compact if it is homeomorphic to a weakly compact subset of some Banach space and it is called Radon-Nikodým compact if it is homeomorphic to a weak-* compact subset of the dual of an Asplund space. By the factorization result of [1], every Eberlein compact space is homeomorphic to a weakly compact subset of a reflexive Banach space, therefore an Eberlein compact is a Radon-Nikodým compact space. However, these two classes are different; indeed any scattered compact space is Radon-Nikodým and any separable, non metrizable scattered compact space cannot be an Eberlein compact since for the class of Eberlein compacta, separability and metrizability are equivalent properties.

The class of Radon-Nikodým compacta has been investigated by several authors [14, 15, 19, 22] after the systematic study made by I. Namioka in [14]. In that paper it is asked:

Problem 4: Find conditions for a Radon-Nikodým compact space to be Eberlein compact.

An answer to this problem was given by [19] and [22] showing that a necessary and sufficient condition for a Radon-Nikodým compact space to be an Eberlein compact is that it is a Corson compact. Recall that a compact space is called Corson compact if it is homeomorphic to a subset of the Σ-product space

$$\Sigma(\Gamma) = \{x \in [-1,1]^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

It is our aim here to give another necessary and sufficient condition on a Radon-Nikodým compact space for it to be Eberlein compact.

If a Radon-Nikodým compact lives in a separable dual, it is metrizable and so it is Eberlein compact. In the non separable case, we know that it lives in a dual of an Asplund space where we can define a projectional resolution of the identity, [6]. These projections are not, in general, weak-* continuous, but if this were the case, we could construct a weak-* to weak continuous injection into a $c_0(\Gamma)$ space, and so the compact space would be Eberlein compact [4, 18, 23].

We shall formulate here a "linking condition" that relates the separable pieces of a given Radon-Nikodým compact space with the separable pieces of the dual norm of the space where it lives, which will be necessary and sufficient for the Radon-Nikodým compact to be Eberlein compact. This condition goes back to the transfer techniques developed in [11] for renormings, and that we have studied in the non metric case in [18].

To formulate our main results we shall need the following:

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Definition 0.1. 1) *Let X be a set and τ_1, τ_2 be two topologies on it. We shall say that X has $\mathcal{L}(\tau_1, \tau_2)$ if for any $x \in X$ there exists a countable set $S(x)$ containing x so that if $A \subset X$ then $\overline{A}^{\tau_2} \subset \overline{\cup\{S(x); x \in A\}}^{\tau_1}$.*

2) *Let (X, τ) be a topological space. We shall say that X has the Linking Separability Property (LSP, for short) if there exists a metric d defined on X, with the metric topology finer than τ, such that X has $\mathcal{L}(d, \tau)$.*

In [18] we studied LSP topological spaces and we shall point out some of their properties when needed.

Our main results are the following.

Theorem A *Let (K, τ) be a compact Hausdorff space. The following are equivalent:*

i) *K is Eberlein compact.*
ii) *There exists a lower semi-continuous metric ϱ on K such that K has $\mathcal{L}(\varrho, \tau)$.*

Theorem B *Let K be a Radon-Nikodým compact space. Then K is Eberlein compact if, and only if, K has LSP.*

As a corollary of the previous results we obtain the following [7, 19, 22]:

Theorem C *Let X be an Asplund generated Banach space, i.e., (there exists an Asplund space E and a map $T : E \to X$ with $\overline{T(E)}^{\|\cdot\|} = X$).*

Then X is WCG if, and only if (B_{X^*}, w^*) has LSP.

For further references on this topic we refer the reader to [5], Chapter 8.

1. Characterizing Eberlein compact spaces.

In this section we shall give the proof of Theorem A. A first step should be to prove that for K verifying condition ii) in Theorem A, K must be a Corson compact (Th. 1.6). To do so we shall need some lemmas. Let us begin by setting some notation.

In this paper we will study compact Hausdorff spaces (K, τ) that admit a lower semi-continuous metric ϱ such that K has $\mathcal{L}(\varrho, \tau)$. We should notice that if this is the case, by a result of Jayne, Namioka and Rogers in [8], the metric topology must be finer than τ, which we will denote by $\tau \preceq \varrho$. In the same paper they state the following result which improves a result by Ghoussoub and Maurey.

Let K be a compact Hausdorff space and let ϱ be a bounded lower semi-continuous metric on K. Then there is a dual Banach space E^* and a homeomorphism $\varphi : K \to E^*$ taken with its $weak^*$ topology, with

$$\|\varphi(x) - \varphi(y)\|_{E^*} = \varrho(x, y) \text{ for all } x, y \in K.$$

The space E is the space of all continuous real-valued functions f on K that satisfy a uniform Lipschitz condition of order 1 with respect to ϱ. Then $\|f\|_{Lip}$ is defined to be the least constant $M > 0$ such that

$$|f(z_1) - f(z_2)| \leq M\varrho(z_1, z_2), \text{ for all } z_1, z_2 \in K,$$

is a norm on E.

The norm $\|\cdot\|$ on E is defined by $\|f\| = \max\{\|f\|_{Lip}, \|f\|_\infty\}$.

The map $\varphi : K \to E^*$ is defined as follows. Given $z \in K$, let $\varphi(z)$ be the linear map $\varphi(z) : E \to \mathbb{R}$ in E^* defined by $\varphi(z)(f) = f(z)$. (So φ sends a point in the compact space to its associated Dirac measure in $E^* \supset C(K)^*$). And we have the following $\|\varphi(x) - \varphi(y)\|_{E^*} = \varrho(x, y)$.

If ϱ is not bounded, we could take a homeomorphism $\psi : \mathbb{R} \to (0, 1)$ and consider $d = \psi \circ \varrho$ which would be a bounded lower semi-continuous metric on K.

Lemma 1.1. *(**Main construction.**) Let (K, τ) be a compact Hausdorff space and ϱ be a lower semi-continuous metric on K, such that K has $\mathcal{L}(\varrho, \tau)$. Let $A_0 \subset C(K)$, $M_0 \subset K$, with $|A_0| = |M_0|$. Then there are sets A and M, with the following properties:*

i) *$A_0 \subset A \subset C(K)$, A is a $\mathbb{Q}$-linear algebra with $\mathbf{1} \in A$, $|A| = |A_0|$.*
ii) *$M_0 \subset M \subset K$, $|M_0| = |M|$.*
iii) *$A \cap B_E$ is a norming set for span $\{\varphi(S(M))\} \subset E^*$ and a norming set for span $\{\overline{\varphi(M)}^{\sigma(E^*,E)}\} \subset E^*$. (Where $S(M)$ is the set associated to M by the property $\mathcal{L}$.)*
iv) *If x and y are in $\overline{M}$, $x \neq y$ there is $f \in A$ with $f(x) \neq f(y)$ and for every $f \in A$ there is $\xi(f) \in M$ with $|f(\xi(f))| = \sup\{|f(x)|; x \in K\}$.*

Proof. We shall construct M and A by an "exhaustion argument" of countable type thanks to $\mathcal{L}(\varrho, \tau)$ we have on K.

For $x \in K$, $S(x)$ will be the countable set given by $\mathcal{L}(\varrho, \tau)$ and $S(N) = \cup\{S(x); x \in N\}$.

For any $f \in C(K)$, let $\xi(f) \in K$ so that $|f(\xi(f))| = \max\{|f(x)|; x \in K\}$.

For any subset of K, $N \subset K$, define a subset of E^* by

$$\Phi(N) = \mathbb{Q} - \text{linear span}\{\varphi(S(N))\}.$$

For $y \in \Phi(N)$ consider the countable subset of E

$$\{f_y^n \in B_E; \|y\|_{E^*} = \text{sup } \{|f_y^n(x)|; n \in \mathbb{N}\}\}.$$

Finally set

$$\Psi(N) = \cup\{f_y^n; n \in \mathbb{N}, y \in \Phi(N)\}.$$

Consider $M_0 \subset K$, $A_0 \subset C(K)$ and define

$$A_1 = \mathbb{Q} - \text{linear algebra generated by } \{\mathbf{1}, \Psi(M_0), A_0)\} \subset C(K),$$

and $M_1 = M_0 \cup \{\xi(f); f \in A_1\}$. It is clear that $|A_1| = |A_0|$, $|M_1| = |M_0|$ and $A_1 \cap B_E$ is a norming set for the $\mathbb{Q}$-linear span $\{\varphi(S(M_0))\}$.

Assume we have defined sequence of sets $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n$ and $M_0 \subset M_1 \subset \ldots M_n$ as A_1 and M_1 above.

Define $A_{n+1} = \mathbb{Q} -$ linear algebra generated by $\{1, \Psi(M_n), A_n\}$ and the set $M_{n+1} = M_n \cup \{\xi(f); f \in A_{n+1}\}$.

Take $A = \cup\{A_n; n \in \mathbb{N}\}$ and $M = \cup\{M_n; n \in \mathbb{N}\}$. Let us show that M and A are the sets we are looking for:

(i) and (ii) are quite clear by construction and since for any point x the set $S(x)$ is at most countable.

By construction, $A \cap B_E$ is norming for span $\{\varphi(S(M))\} \subset E^*$. Thus, $A \cap B_E$ norms

$$\text{span } \overline{\varphi(S(M))}^{\|\cdot\|^*} \subset \overline{\text{span } \varphi(S(M))}^{\|\cdot\|^*}.$$

Now by $\mathcal{L}(\varrho, \tau)$, $\overline{\varphi(M)}^{w^*} \subset \overline{\varphi(S(M))}^{\|\cdot\|^*}$ and that implies that $A \cap B_E$ norms span $\overline{\{\varphi(M)\}}^{w^*}$.

Let us check iv). Take $x, y \in \overline{M}$, $x \neq y$, and assume that for all $f \in A$ we had $f(x) = f(y)$. Since φ injects K homeomorphically in E^*, we would have $\varphi(x) \neq \varphi(y)$. Now since they belong to $K = \overline{M}^\tau$, there must be $(x_n) \in S(M)$ and $(y_n) \in S(M)$ converging to x and y in ϱ distance by $\mathcal{L}(\varrho, \tau)$.

Let us fix $n \in \mathbb{N}$. There must be $p \in \mathbb{N}$, such that $x_n, y_n \in S(M_p)$ (since $S(M_j)$ is an increasing sequence), therefore

$$\varphi(x_n) - \varphi(y_n) \in \mathbb{Q} - \text{linear span}\{\varphi(S(M_p))\} \subset E^*$$

whose members are normed in $\Psi(M_p) \subset A_{p+1} \subset A$. The same argument holds for any $n \in \mathbb{N}$ and so we have $\varphi(x_n) - \varphi(y_n)$ is normed in $A \cap B_E$ for any $n \in \mathbb{N}$. Finally we have

$$\varrho(x_n, y_n) = \|\varphi(x_n) - \varphi(y_n)\|_{E^*} = \sup\{|f(\varphi(x_n) - \varphi(y_n))|; f \in A \cap B_E\} \leq$$
$$\leq \sup\{|f(\varphi(x_n) - \varphi(x)| + |f(\varphi(x) - \varphi(y))| + |f(\varphi(y) - \varphi(y_n))|; f \in A \cap B_E\} \leq$$
$$\leq \|\varphi(x_n) - \varphi(x)\|_{E^*} + \|\varphi(y) - \varphi(y_n)\|_{E^*} = \varrho(x_n, x) + \varrho(y_n, y)$$

and that implies that $\lim_{n\to\infty} \varrho(x_n, y_n) = 0$ hence $x = y$ wich contradicts the hypothesis.

The second part of iv) is clear by construction. ■

Lemma 1.2. *For sets A and M as in Lemma 1.1, there exists a norm-one projection $P : C(K) \to C(K)$ with:*

i) $P(C(K)) = \overline{A}^{\|\cdot\|_\infty}$.
ii) *P is an homomorphism of algebras with P(**1**)=**1**.*
iii) *There is a continuous retraction $r : K \to \overline{M}$ such that P(f)=$f \circ r$ for all $f \in C(K)$.*
iv) $\varrho(r(x), r(y)) \le \varrho(x, y)$ *for all $x, y \in K$.*

Proof. Let us consider $C(\overline{M})$ with its supremum norm $|||\cdot|||$ and let R be the restriction map $R : C(K) \to C(\overline{M})$, $R(f) = f_{|\overline{M}}$.

Given $\varepsilon > 0$, and $f \in \overline{A}^{\|\cdot\|_\infty}$ there exists $g \in A$ with $\|g - f\|_\infty < \varepsilon$. Let $\xi(g) \in M$ with $|g(\xi(g))| = \|g\|_\infty$. Then we have:

$$\|f\|_\infty \le \|f - g\|_\infty + \|g\|_\infty \le \varepsilon + |g(\xi(g))| = |||Rg||| + \varepsilon \le$$
$$\le |||Rg - Rf||| + |||Rf||| + \varepsilon \le |||Rf||| + 2\varepsilon$$

Since the reasoning is valid for every $\varepsilon > 0$ we should have $\|f_\infty\| \le |||Rf|||$ for all $f \in \overline{A}^{\|\cdot\|_\infty}$ and R is an isometry and algebraic homomorphism between $\overline{A}^{\|\cdot\|_\infty}$ and $(C(\overline{M}), |||\cdot|||)$. Since A separates the points of $\overline{M}$ and contains **1**, $R(\overline{A}^{\|\cdot\|_\infty})$ must coincide with $C(\overline{M})$ by the Stone-Weierstrass theorem. Then

$$R^{-1} : C(\overline{M}) \to \overline{A}^{\|\cdot\|_\infty} \hookrightarrow C(K)$$

should be a linear extension operator and the projection P is defined by $P = R^{-1} \circ R$ and it obviously verifies i) and ii).

iii) follows from a very special case of variants of theorems of Banach-Stone and Gelfand-Naimark. Indeed every measure δ_x for $x \in K$ give us a character for the algebra $C(K)$; i.e., a linear functional multiplicative sending **1** to **1**, and every character is a Dirac measure. Any algebraic homomorphism and linear isometry between algebras puts in one-to-one correspondence the characters of the algebras by the transpose isomorphism. Dealing with the weak* topology we should have consequently, that for every $x \in K$, δ_x provides a character for the algebra A which corresponds with a Dirac measure $\delta_{r(x)} \in \overline{M}$. See [21]. This provides us with a continuous retraction $r : K \to \overline{M}$ and $P(f) = f \circ r$ since $f \circ r$ is continuous on K and $f \circ r_{|\overline{M}} = f_{|\overline{M}}$.

Let us finish proving iv). For x and y in K we have $r(x) \in \overline{M}$ and $r(y) \in \overline{M}$, so

$$\varphi(r(x)) - \varphi(r(y)) \in \text{span}\,\{\overline{\varphi(M)}^{\sigma(E^*,E)}\}$$

and by iii) in Lemma 1.1 we have

$$\varrho(r(x), r(y)) = \|\varphi(r(x)) - \varphi(r(y))\|_{E^*} =$$
$$= \sup_{f\in A\cap B_E} \{| < \varphi(r(x)) - \varphi(r(y)), f > |\} = \sup_{f\in A\cap B_E} \{|f(r(x)) - f(r(y))|\} =$$
$$= \sup_{f\in A\cap B_E} \{| < f \circ r, \delta_x - \delta_y > |\} = \sup_{f\in A\cap B_E} \{| < P(f), \delta_x - \delta_y > |\} \le$$
$$\le \sup_{f\in B_E} \{| < f, \delta_x - \delta_y > |\} = \sup_{f\in B_E} \{| < f, \varphi(x) - \varphi(y) > |\} =$$
$$= \|\varphi(x) - \varphi(y)\|_{E^*} = \varrho(x, y).$$

■

Proposition 1.3. *Let (K, τ) a compact Hausdorff space and ϱ be a lower semi-continuous metric on it such that K has $\mathcal{L}(\varrho, \tau)$. Then:*

$$dens(K, \tau) = dens(K, \varrho) = dens(C(K), \|\cdot\|_\infty).$$

Proof. It is very clear from the definition of the property $\mathcal{L}(\varrho,\tau)$, that $dens(K,\tau) = dens(K,\varrho)$. Since $dens(K,\tau) \le dens(C(K), \|\cdot\|_\infty)$ always holds, we only have to show that $dens(C(K), \|\cdot\|_\infty) \le dens(K,\tau)$.

Now let $M_0 = \{x_\alpha; 0 \le \alpha < \mu\}$ be a dense subset of K, where μ is the first ordinal number whose cardinality $|\mu| = \text{dens}(K,\tau)$. And let A_0 be any subset of $C(K)$ of the same cardinality than M_0.

Applying Lemma 1.1 and Lemma 1.2 to A_0 and M_0, we obtain $A \supset A_0$ and $M \supset M_0$ with the properties stated in both results. But $\overline{M} = K$ and therefore the restriction R is the identity. So $\overline{A} = C(K)$ and the density character of $\overline{A}$, and hence $C(K)$, is at most the cardinality of M_0. ■

The previous Lemmas can be applied to obtain the following:

Theorem 1.4. *Let (K,τ) be a compact Hausdorff space and ϱ be a lower semi-continuous metric on it with $\mathcal{L}(\varrho,\tau)$.*

Then, there exists $\{P_\alpha; \omega_0 \le \alpha \le \mu\}$ a PRI on $C(K)$, and a family of continuous retractions $r_\alpha : K \to K$ with $P_\alpha(f) = f \circ r_\alpha$, dens($r_\alpha(K)) \le |\alpha|$ and $\varrho(r_\alpha(x), r_\alpha(y)) \le \varrho(x,y)$ for all $x,y \in K$ and for all $\alpha \in [\alpha,\mu]$.

Moreover, $r_\alpha \to r_\beta$ for $\alpha \to \beta$ pointwise on K in the ϱ topology. The latter implies that given $x \in K$ and $\varepsilon > 0$, the set

$$\{\alpha; \omega_0 \le \alpha \le \mu, \varrho(r_{\alpha+1}(x), r_\alpha(x)) > \varepsilon\}$$

is finite. Thus, the set $\{\alpha; \omega_0 \le \alpha \le \mu, r_{\alpha+1}(x) \ne r_\alpha(x)\}$ is at most countable.

Proof. Let $|\mu|$ be the first ordinal number such that $|\mu| = dens(C(K))$ and let $\{x_\alpha; 0 \le \alpha < \mu\}$ and $\{f_\alpha; 0 \le \alpha < \mu\}$ be dense subsets of K and $C(K)$ respectively.

Let us begin by applying Lemma 1.1 and Lemma 1.2 to the sets $A_0 = \{f_\alpha; 0 \le \alpha \le \omega_0\}$ and $M_0 = \{x_\alpha; 0 \le \alpha \le \omega_0\}$. We obtain $A_{\omega_0} = A$, M_{ω_0} and P_{ω_0} with the properties stated in both Lemmas.

Now let $\beta \le \mu$ be any ordinal number and assume that for any $\alpha < \beta$, we have constructed families $A_{\omega_0} \subset \ldots \subset A_\alpha$ of $\mathbb{Q}$-algebras and $M_{\omega_0} \subset \ldots \subset M_\alpha \subset K$, with $S(M_\alpha) \subset M_{\alpha+1}$ as well as the corresponding linear projections $\{P_\alpha; \omega_0 \le \alpha < \beta\}$ satisfying the conditions in both Lemmas and $|\alpha| = |M_\alpha| = |A_\alpha|$.

If β is not a limit ordinal, i.e., $\beta = \alpha + 1$, set

$$A_0 = A_\alpha \cup \{f_{\alpha+1}\} \text{ and } M_0 = S(M_\alpha \cup \{x_{\alpha+1}\}).$$

Apply the Lemmas to these sets to obtain $A_{\alpha+1}$ and $M_{\alpha+1}$ satisfying all the conditions required.

If β is a limit ordinal define

$$A_\beta = \cup\{A_{\alpha+1}; \omega_0 \le \alpha < \beta\}, M_\beta = \cup\{M_{\alpha+1}; \omega_0 \le \alpha < \beta\}.$$

We shall see now that A_β and M_β satisfy the conditions in Lemma 1.1.

First let us show that $A_\beta \cap B_E$ norms span$\{\varphi(S_{M_\beta})\} \subset E^*$.

Take $x \in$ span $\varphi(S(M_\beta))$, then x is a finite linear combination of points in $\cup\{\varphi(S(M_{\alpha+1})) : \omega_0 \le \alpha < \mu\}$. Hence, by construction, there must be α such that $x \in$ span $\{\varphi(S(M_\alpha))\}$ which is normed, by induction hypothesis, by $A_\alpha \cap B_E$, which is contained in $A_\beta \cap B_E$.

Consequently as in the Lemma, we will also have that $A_\beta \cap B_E$ norms span$\overline{\{\varphi(S(M_\beta))\}}^{w^*} \subset E^*$. Since $A_\beta \cap B_E$ norms

$$\text{span } \overline{\varphi(S(M_\beta))}^{\|\cdot\|^*} \subset \overline{\text{span } \varphi(S(M_\beta))}^{\|\cdot\|^*}.$$

Now by the $\mathcal{L}(\varrho,\tau)$, $\overline{\varphi(M_\beta)}^{w^*} \subset \overline{\varphi(S(M_\beta))}^{\|\cdot\|^*}$ and that implies that $A_\beta \cap B_E$ norms span $\overline{\{\varphi(M_\beta)\}}^{w^*}$.

To prove iv) we essentially have to follow the proof of Lemma 1.1. Take $x,y \in \overline{M_\beta}$, $x \ne y$, and assume that for all $f \in A$ we had $f(x) = f(y)$. Then we would have $\varphi(x) \ne \varphi(y)$. Now since they belong to $K_\beta = \overline{M_\beta}^{\tau}$, there must be $(x_n) \in S(M_\beta)$ and $(y_n) \in S(M_\beta)$ converging to x and y in ϱ distance by $\mathcal{L}(\varrho, \tau)$.

Let us fix $n \in \mathbb{N}$. There must be $\alpha(n) < \beta$, such that $x_n, y_n \in S(M_{\alpha(n)})$ (since $S(M_\alpha)$ is an increasing sequence), therefore

$$\varphi(x_n) - \varphi(y_n) \in \mathbb{Q} - \text{linear span}\{\varphi(S(M_{\alpha(n)}))\} \subset E^*$$

whose members are normed in $A_{\alpha(n)} \subset A_\beta$. The same argument holds for any $n \in \mathbb{N}$ and so we have $\varphi(x_n) - \varphi(y_n)$ are normed in $A_\beta \cap B_E$ for any $n \in \mathbb{N}$. And, as in the Lemma, we would get $x = y$. The second part of iv) in Lemma 1.1 is clear.

Consequently we will have by Lemma 1.2 a projection P_β, with the range of P_β equal to $\overline{A}^{\|\cdot\|_\infty}$, and a continuous retraction with $r_\beta(K) = \overline{M_\beta}$ and $\text{dens}(r_\beta(K)) \leq |\beta|$.

To finish let us show that for each $x \in K$, $r_\alpha(x) \to r_\beta(x)$ in the ϱ topology.

Since $S(M_\alpha) \subset M_{\alpha+1}$ for any α, we should have that for any β limit ordinal,

$$\overline{M_\beta}^{\tau} \subset \overline{\cup_{\alpha<\beta} M_{\alpha+1}}^{\varrho} \subset \overline{M_\beta}^{\beta}$$

therefore $\overline{M_\beta}^{\tau} = \overline{M_\beta}^{\varrho}$.

Trivially, $r_\alpha(x) \to r_\beta(x)$ for any $x \in M_\beta$. Since $\{r_\alpha\}$ are ϱ-uniformily equicontinuous, and $\overline{M_\beta}^{\tau} = \overline{M_\beta}^{\varrho}$, we have $r_\alpha(x) \to r_\beta(x)$ for all $x \in \overline{M_\beta}^{\tau}$. ■

The following result is in [18].

Remark 1.5. *Let (X, τ) be a LSP topological space, then any subspace of X is also LSP. In fact if d is a metric on X such that X has $\mathcal{L}(d, \tau)$ and $H \subset X$ then H has $\mathcal{L}(d, \tau)$.*

Theorem 1.6. *Let (K, τ) be a compact Hausdorff space and ϱ be a lower semi-continuous metric on it with $\mathcal{L}(\varrho, \tau)$. Then K is a Corson compact.*

Proof. We are going to show it by induction on the density character of the compact.

If (K, τ) is separable, by Proposition 1.7, it is metrizable, hence Corson compact.

Now let μ be the first ordinal with cardinality equal to $\text{dens}(K, \tau)$, and assume that for any compact space of density character less than $|\mu|$ and having LSP for a lower semi-continuous metric is Corson.

Let $\{r_\alpha : \omega_0 \leq \alpha < \mu\}$ be the family of retractions on K given by Theorem 1.4.

Let $K_\alpha = r_\alpha(K) \subset K$. By the construction $\text{dens}(K_\alpha) \leq |\alpha|$. Since property $\mathcal{L}$ is hereditary (Remark 1.5), by the induction hypothesis each K_α is a Corson compact. Hence, for any α, $\omega_0 \leq \alpha < \mu$ there exists a set Γ_α, and a homeomorphism $\Psi_\alpha : K_\alpha \to \Sigma(\Gamma_\alpha) \subset \mathbb{R}^{\Gamma_\alpha}$.

Let Γ be the disjoint union of the sets $\{\Gamma_\alpha\}_{\omega_0 < \alpha < \mu}$ and $\mathbb{N}$, and define $T : K \to \mathbb{R}^\Gamma$ by

$$T(x)(n) = \Psi_{\omega_0}(r_{\omega_0}(x))(n), n \in \mathbb{N}$$

$$T(x)(\gamma) = (\Psi_{\alpha+1}(r_{\alpha+1}(x))(\gamma) - \Psi_{\alpha+1}(r_\alpha(x))(\gamma)), \text{ for } \gamma \in \Gamma_{\alpha+1}.$$

Given $x \in K$ since the set $\{\alpha; r_{\alpha+1}(x) \neq r_\alpha(x)\}$ is at most countable and $\Psi_\alpha(r_\alpha(x))$ lives in $\Sigma(\Gamma_\alpha)$ for any α, it clearly follows that $T(x)$ lives in $\Sigma(\Gamma)$.

T is clearly continuous. To see that it is an injection, let us take $x, y \in K$ and suppose $T(x) = T(y)$. Let us show that $r_\alpha(x) = r_\alpha(y)$ for all α which would imply $x = y$.

In particular, $\Psi_{\omega_0}(r_{\omega_0}(x)) = \Psi_{\omega_0}(r_{\omega_0}(y))$, and since Ψ_{ω_0} is one-to-one, $r_{\omega_0}(x) = r_{\omega_0}(y)$. So assume $r_\alpha(x) = r_\alpha(y)$ for all $\alpha < \beta$. Since $r_\alpha(x) \to r_\beta(x)$ we would obtain $r_\beta(x) = r_\beta(y)$. (Now $x = r_\mu(x) = r_\mu(y) = y$). (For non limit ordinals is also trivial).

Hence T injects homeomorphically (K, τ) into a sigma product. Thus, K is a Corson compact. ■

The conditions on the following two propositions are clearly fulfilled if K has $\mathcal{L}(\varrho, \tau)$.

Proposition 1.7. *Let (K, τ) be a compact Hausdorff space and ϱ a lower semi-continuous metric on K. If every separable subset of K is ϱ-separable too, then the separable subsets of K are metrizable.*

Proof. Since the ϱ-topology is finer than τ, the result follows from the fact that any compact image of a separable metrizable space is metrizable ([3], Theorem 3.1.20). ∎

It is known after Namioka [14], that a compact space is Radon-Nikodým compact if and only if it is fragmented by a lower semi-continuous metric. Recall that a topological space is said to be fragmented by a metric if for any $\varepsilon > 0$, and any non-empty subset A of the space, there exists a relatively open subset of A with diameter less than ε.

Proposition 1.8. *Let (K, τ) be a compact Hausdorff space and ϱ a lower semi-continuous metric on K. If every separable subset of K is ϱ-separable too, then ϱ is a fragmenting metric. Hence, K is RN compact.*

Proof. The result follows immediately from Theorem 4.1, equivalence (c) and (j), in [8], where one should consider the irreducible map p. ∎

We can now **prove Theorem A** in the introduction:

i)→ii) K is Radon-Nikodým (Proposition 1.8) and Corson (Theorem 1.6), so by the mentioned result in [19, 22], we conclude that K is Eberlein.

ii)→i) We can see (K, τ) as a weakly compact subset of a WCG Banach space E. In [16] we showed that any WCG Banach space has $\mathcal{L}(\|\cdot\|, weak)$, hence by Remark 1.5 so does K for τ and $\|\cdot\|$. ∎

2. Consequences in Banach spaces.

In order to show Theorem B, we need the following definition by Jayne, Namioka and Rogers [9].

Definition 2.1. *Let (X, τ) be a topological space and d be a metric on X. We shall say that X is σ-fragmented by d if for every $\varepsilon > 0$, it is possible to write $X = \cup_{n=1}^{\infty} X_n^{\varepsilon}$, such that for each $n \in \mathbb{N}$ and any subset $A \subset X_n^{\varepsilon}$ there exists a relatively τ-open subset of A with d-diameter less than ε.*

And also the following result from [18].

Remark 2.2. *Let (X, τ) have LSP and ϱ be any metric on X finer than τ. If (X, τ) is σ-fragmented by ϱ, then X has $\mathcal{L}(\varrho, \tau)$.*

Now let us give the **proof of Theorem B**:

If K is Eberlein the reasoning in the proof of Theorem A applies. So let K have LSP, i.e., there exists a metric on K, say d, with the metric topology finer than τ and such that K has $\mathcal{L}(d, \tau)$.

Since K is Radon-Nikodým, there must be a lower semi-continuous metric ϱ fragmenting (K, τ). Apply Remark 2.2 to obtain K has $\mathcal{L}(\varrho, \tau)$, now Theorem A applies to conclude that K is Eberlein. ∎

We can also extend Theorem 8.3.4 in [5] giving the Banach space version of the former result, i.e., Theorem C in the introduction. The **proof of Theorem C** is as follows.

T^* is one-to-one and gives an homeomorphism between (B_{X^*}, w^*) and $(T^*(B_{X^*}), w^*)$.

If X is WCG we know that (B_{X^*}, w^*) is Eberlein compact and it has LSP.

Conversely, if (B_{X^*}, w^*) has LSP, since it is Radon-Nikodým compact we have, by Theorem B, (B_{X^*}, w^*) is Eberlein. Now Theorem 8.3.4 in [5] applies to give X is WCG. ∎

3. Final remarks.

In [18] we studied the relationship between property $\mathcal{L}$, σ-fragmentability and property SLD of Jayne, Namioka and Rogers. The last property is defined as follows:

Definition 3.1. *We shall say that X has a countable cover by set of small local diameter (SLD) if for every $\varepsilon > 0$ it is possible to write $X = \cup_{n=1}^{\infty} X_n^{\varepsilon}$, such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relatively τ-neighbourhood of d-diameter less than ε.*

It was shown that when (X,τ) is a metric space and ϱ is a metric on X finer than τ, the conditions: X has $\mathcal{L}(\varrho,\tau)$, (X,τ) is ϱ-σ-fragmented and (X,τ) has ϱ-SLD, are all equivalent.

Our aim now is to show that this is no longer true when τ is a non-metrizable topology, i.e., we shall give examples of a space with property LSP and not SLD, and another with SLD and not LSP. First, one more property from [18] is needed:

Remark 3.2. *Let (X,τ) be σ-fragmented by a metric d finer than τ (resp. d-SLD). If ϱ is another metric such that X has $\mathcal{L}(\varrho,\tau)$, then (X,τ) is σ-fragmented by ϱ (resp. ϱ-SLD.)*

Example 3.3. *Let (K,τ) be a separable non metrizable RN compact, then K does not have property LSP.*

Proof. If there were a metric ϱ finer than the topology of K, with $\mathcal{L}(\varrho,\tau)$, since K is RN, i.e., fragmented by a lower semi-continuous metric, then by Remark 2.2, K would have property $\mathcal{L}$ for this metric too, therefore by Proposition 1.7, K would be metrizable. ■

The next example is due to A. Moltó, and can be found in [2].

Example 3.4. *There exists a compact Hausdorff space (K,τ) and a metric ϱ such that (K,τ) has the ϱ-SLD property and it fails to have $\mathcal{L}(\varrho,\tau)$. Moreover, (K,τ) has not the LSP.*

Proof. We denote by $\Delta = \{0,1\}^{\mathbb{N}}$ the Cantor set, and by $\mathcal{D}$ the set of finite sequences of 0's and 1's. For $\sigma \in \mathcal{D}$, we denote by I_σ the clopen (i.e. closed and open) subset of Δ consisting of those sequences which start with σ. We consider the following set K_0 of functions on Δ: the set K_0 contains the characteristic functions of the sets I_σ, $\sigma \in \mathcal{D}$ (denoted by χ_{I_σ}); of the points of Δ, and the function identically equal to zero, denoted by ϕ.

When eqquiped with the topology of pointwise convergence on Δ, K_0 becomes a compact set, which is separable, scattered, nonmetrizable and $K_0^{(3)} = \emptyset$.

By a result of Deville $C(K)^*$ admits an equivalent dual LUR norm, which is equivalent ([20]) to have $(C(K)^*, w^*)$ the $\|\cdot\|^*$-SLD property.

So (K,τ) has ϱ-SLD for a τ-lower semi-continuous metric, (ϱ is the restriction to K of the dual norm). Now, if K had $\mathcal{L}(\varrho,\tau)$, by Proposition 1.7, (K,τ) would be metrizable (since it is separable) which is not true.

To prove the moreover part, we only have to apply Remark 2.2. ■

So Example 3.4 shows that for a compact space (K,τ) that has ϱ-SLD property we may not have LSP (not only $\mathcal{L}(\varrho,\tau)$).

Remark 3.5. *In* [16] *we proved that under CH, ℓ^∞ had $\mathcal{L}(\|\cdot\|, weak)$ and it has not SLD* [10].

The same arguments as in the example above, work for the next result.

Proposition 3.6. *Let K be a scattered compact space with $K^{(\omega_1)} = \emptyset$, having separable subsets which are non-metrizable. Then K has ϱ-SLD property for a lower semi-continuous metric and K does not have the LSP.*

Example 3.7. *(B_{ℓ^∞}, w^*) is a metrizable compact space, and B_{ℓ^∞} has not $\mathcal{L}(\|\cdot\|_\infty, w^*)$.*

Proof. It is clear since (ℓ^∞, w^*) is separable whereas $(\ell^\infty, \|\cdot\|_\infty)$ is not. (Of course (ℓ^∞, w^*) lacks the $\|\cdot\|_\infty$-SLD property [9]).

REFERENCES

[1] W. J. Davies, T. Figiel, W. B. Johnson and A. Pełcziński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311-327.

[2] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Vol. 64, Pitman Monographs and Surveys in Pure and Applied Mathematics, 1993.

[3] R. Engelking, *General Topology*, Monographie Mat., Vol. 60, Polish Scientific, Warsaw, 1977.

[4] M. Fabian, *Each weakly countably determined Asplund space admits a Fréchet differentiable norm*, Bull. Australian Math. Soc. **35** (1987), 363-372.

[5] M. Fabian, *Gâteaux Differentiability of Convex Functions and Topology. Weak Asplund Spaces*, Canadian Math. Soc. Series of Monog. and Advanced Texts, Wiley, 1997.

[6] M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), 141-151.

[7] M. Fabian and J. H. Whitfield, *On equivalent characterization of weakly compactly generated Banach spaces*, Rocky Mountains J. Math. **24** (1994), 1363-1378.

[8] J. E. Jayne, I. Namioka, C. A. Rogers, *Norm fragmented weak* compact sets*, Collect. Mat. **41** (1990), 133-163.

[9] J. E. Jayne, I. Namioka, C. A. Rogers, *σ-fragmentable Banach spaces*, Mathematika **39** (1992), 161-188 and 197-215.

[10] J. E. Jayne, I. Namioka, C. A. Rogers, *Topological properties of Banach spaces*, Proc. London Math. Soc. (3) 66 (1993), 651-672.

[11] A. Moltó, J. Orihuela, S. Troyanski, *Locally uniform rotund renorming and fragmentability*, Proc. London Math. Soc. (3) 75 (1997), 619-640.

[12] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia, *On Weakly Locally Uniformly Rotund Banach Spaces*, J. Funct. Anal. 163 (1999), no.2, 252-271.

[13] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia, *Non-linear transfer technique*, Preprint.

[14] I. Namioka, *Radon-Nikodým compact spaces and fragmentability*, Mathematika 34 (1989), 258-281.

[15] I. Namioka, *Eberlein and Radon-Nikodým compact spaces*, Lecture Notes at University College London, Autumn 1985.

[16] L. Oncina, *Descriptive Banach spaces and Eberlein compacta*, Doctoral Thesis, Universidad de Murcia, 1999.

[17] L. Oncina, *The JNR property and the Borel structure of a Banach space*, Serdica Math. J., vol. 26 (1) (2000) 13-32.

[18] L. Oncina, *On WCG Asplund Banach spaces*, Preprint.

[19] J. Orihuela, W. Schachermayer, M. Valdivia, *Every Radon-Nikodým Corson compact space is Eberlein compact*, Studia Mathematica 98 (2) (1991).

[20] M. Raja, *Borel measurability and renorming in Banach spaces*, Tesis Doctoral, Universidad de Murcia, 1999.

[21] C. Stegall, *A proof of the theorem of Amir and Lindenstrauss*, Israel J. Math. 68 (1989), no. 2, 185-192.

[22] C. Stegall, *More facts about conjugate Banach spaces with the Radon-Nikodým property II*, Acta Univ. Carolin.-Math. Phys. 32 (1991), 47-54.

[23] M. Valdivia, *Resolution of the identity in certain Banach spaces*, Collect. Math. **39** (1988), 127-140.

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30.100 ESPINARDO, MURCIA, SPAIN

E-mail address: luis@um.es